

Møller's Energy in the Kantowski-Sachs Space-time

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Abstract

We present a counter example to paper [1] and show that the result obtained is correct for a class of metric but not general. We calculate the total energy of the Kantowski-Sachs space-time by using the energy-momentum definitions of Møller in the theory of general relativity and the tetrad theory of gravity.

1 Introduction

Since the birth of the theory of general relativity and this theory has been accepted as a superb theory of space-time and gravitation, as many physical aspects of nature have been experimentally verified in this theory. However, this theory is still incomplete theory, namely, it lacks definition of energy and momentum. In this theory many physicist have introduced different types of energy-momentum complexes [2], each of them being a pseudo-tensor, to solve this problem. The non-tensorial property of these complexes is inherent in the way they have been defined and so much so it is quite difficult to conceive of a proper definition of energy and momentum of a given system. The recent attempt to solve this problem is to replace the theory of general relativity by another theory, concentrated on the gauge theories for the translation group, the so called teleparallel equivalent of general relativity. We were hoping that the theory of teleparallel gravity would solve this problem. Unfortunately, the localization of energy and momentum in this theory is still an open, unresolved and disputed problem as in the theory of general relativity.

Møller modified the theory of general relativity by constructing a gravitational theory based on Weitzenböck space-time. This modification was to overcome the problem of the energy-momentum complex that appears in Riemannian space. In a series of paper [3]-[5], he was able to obtain a general expression for a satisfactory energy-momentum complex in the absolute

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parallelism space. In this theory the field variable are 16 tetrad components $h_a{}^\mu$, from which the Riemannian metric arises as

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu. \quad (1.1)$$

The basic purpose of this paper is to obtain the total energy of the Kantowski-Sachs space-time by using the energy-momentum definitions of Møller in the theory of general relativity and the tetrad theory of gravity.

The standard representation of Kantowski and Sachs space-times are given by [7]

$$ds^2 = dt^2 - A^2(t)dr^2 - B^2(t)(d\theta^2 + \sin^2\theta d\phi^2), \quad (1.2)$$

where the functions $A(t)$ and $B(t)$ are function in t and determined from the field equations.

For more detailed descriptions of the geometry and physics of this space-time see for example [7], [8] and [9].

2 On the fourth component of Einstein's complex

Prasanna have shown that space-times with purely time dependent metric potentials have their components of total energy and momentum for any finite volume $(T_i^4 + t_i^4)$ identically zero. He had used the Einstein complex for the general Riemannian metric

$$dS^2 = g_{ij}(x^0)dx^i dx^j, \quad (2.1)$$

and concluded the following: For space-times with metric potentials g_{ij} being functions of time variable alone and independent of space variable the components $(T_i^4 + t_i^4)$ vanish identically as a consequence of conservation law.

Unfortunately the conclusion above is not the solution to the problem considered, in the sense that it does not give the same result for all metrics have form (2.1), using Einstein complex. If (2.1) is given in spherical coordinates, then Prasanna's conclusion is correct by using Møller's complex but not correct for all metrics by using Einstein's complex. Because Møller's complex could be utilized to any coordinate system, but Einstein's complex give meaningful result if it is evaluated in Cartesian coordinates. In the present paper we have found that the total energy for the Kantowski-Sachs

space-time is identically zero by using Møller's complex, but not zero by using Einstein's complex.

In a recent paper [10], Gad and Fouad have found the energy and momentum distribution of Kantowski-Sachs space-time, using Einstein, Bergmann-Thomson, Landau-Lifshitz and Papapetrou energy momentum complexes. In this section we restrict our attention to the Einstein's complex which is defined by [11]

$$\theta_i^k = T_i^k + t_i^k = u_{i,k}^{[jk]}, \quad (2.2)$$

with

$$u_{i,k}^{[jk]} = \frac{1}{\kappa} \frac{g_{in}}{\sqrt{-g}} [-g(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}. \quad (2.3)$$

The energy and momentum in the Einstein's prescription are given by

$$P_i = \int \int \int \theta_i^0 dx^1 dx^2 dx^3. \quad (2.4)$$

The Einstein energy-momentum complex satisfies the local conservation law

$$\frac{\partial \theta_i^k}{\partial x^k} = 0. \quad (2.5)$$

The energy density for the space-time under consideration, in the Cartesian coordinates, obtained in [10] is

$$\theta_0^0 = \frac{1}{8\pi A r^4} (A^2 r^2 - B^2), \quad (2.6)$$

and the total energy is

$$E_{Ein} = P_0 = \frac{1}{2Ar} (A^2 r^2 + B^2),$$

Following the approach in [10], we obtain the following components of θ_0^k

$$\begin{aligned} \theta_0^1 &= -\frac{x}{8\pi A^2 r^4} [\dot{A} A^2 r^2 - B(2A\dot{B} - B\dot{A})], \\ \theta_0^2 &= -\frac{y}{8\pi A^2 r^4} [\dot{A} A^2 r^2 - B(2A\dot{B} - B\dot{A})], \\ \theta_0^3 &= -\frac{z}{8\pi A^2 r^4} [\dot{A} A^2 r^2 - B(2A\dot{B} - B\dot{A})]. \end{aligned} \quad (2.7)$$

The components (2.6) and (2.7) satisfy the conservation law (2.5).

Hence from equations (2.6) and (2.7), we have $\theta_0^i = T_0^i + t_0^i \neq 0$ consequently $\theta_0^0 = T_0^0 + t_0^0$ is not identically zero.

3 Energy in the theory of General Relativity

In the general theory of relativity, the energy-momentum complex of Møller in a four dimensional background is given as [3]

$$\mathfrak{S}_i^k = \frac{1}{8\pi} \chi_{i,l}^{kl}, \quad (3.1)$$

where the antisymmetric superpotential χ_i^{kl} is

$$\chi_i^{kl} = -\chi_i^{lk} = \sqrt{-g} \left(\frac{\partial g_{in}}{\partial x^m} - \frac{\partial g_{im}}{\partial x^n} \right) g^{km} g^{nl}, \quad (3.2)$$

\mathfrak{S}_0^0 is the energy density and \mathfrak{S}_α^0 are the momentum density components. Also, the energy-momentum complex \mathfrak{S}_i^k satisfies the local conservation laws:

$$\frac{\partial \mathfrak{S}_i^k}{\partial x^k} = 0 \quad (3.3)$$

The energy and momentum components are given by

$$P_i = \int \int \int \mathfrak{S}_i^0 dx^1 dx^2 dx^3 = \frac{1}{8\pi} \int \int \int \frac{\partial \chi_i^{0l}}{\partial x^l} dx^1 dx^2 dx^3. \quad (3.4)$$

For the line element (1.2), the only non-vanishing components of χ_i^{kl} are

$$\begin{aligned} \chi_1^{01} &= -\frac{B^2(t)}{A(t)} \sin \theta, \\ \chi_2^{02} &= -A(t) \sin \theta, \\ \chi_3^{03} &= -\frac{A(t)}{\sin \theta}. \end{aligned} \quad (3.5)$$

Using these components in equation (3.1), we get the energy and momentum densities as following

$$\mathfrak{S}_0^0 = 0. \quad (3.6)$$

$$\mathfrak{S}_1^0 = \mathfrak{S}_3^0 = 0, \mathfrak{S}_2^0 = -A(t) \cos \theta. \quad (3.7)$$

From equation (3.4) and (3.5) and applying the Gauss theorem, we obtain the total energy and momentum components in the following form

$$P_0 = E = 0, \quad (3.8)$$

$$P_\alpha = 0, \quad (3.9)$$

4 Energy in the Tetrad Theory of gravity

The super-potential of Møller in the tetrad theory of gravity is given by (see [4]- [6])

$$U_a^{bc} = \frac{\sqrt{-g}}{2\kappa} P_{efh}^{dbc} [\Phi^f g^{eh} g_{ad} - \lambda g_{ad} \gamma^{efh} - (1 - 2\lambda) g_{ad} \gamma^{hfe}], \quad (4.1)$$

where

$$P_{efh}^{dbc} = \delta_e^d g_{fh}^{bc} + \delta_f^d g_{he}^{bc} - \delta_h^d g_{ef}^{bc},$$

with g_{fh}^{bc} being a tensor defined by

$$g_{fh}^{bc} = \delta_f^b \delta_h^c - \delta_h^b \delta_f^c,$$

γ_{abc} is the con-torsion tensor given by

$$\gamma_{abc} = h_{i\mu} h^i{}_{\nu;\rho} \quad (4.2)$$

and Φ_a is the basic vector defined by

$$\Phi_a = \gamma^b{}_{ab}.$$

The energy in this theory is expressed by the following surface integral

$$E = \lim_{r \rightarrow \infty} \int_{r=const.} U_0^{0\alpha} n_\alpha dS, \quad (4.3)$$

where n_α is the unit three vector normal to the surface element dS .

The tetrad components of the space-time (1.2), using (1.1), are as following

$$\begin{aligned} h^a{}_\mu &= [1, A(t), B(t), B(t) \sin \theta], \\ h_a{}^\mu &= [1, A^{-1}(t), B^{-1}(t), \frac{B^{-1}(t)}{\sin \theta}]. \end{aligned} \quad (4.4)$$

Using these components in (4.2), we get the non-vanishing components of $\gamma_{\mu\nu\beta}$ as following

$$\begin{aligned} \gamma_{011} = -\gamma_{101} &= -A(t) \dot{A}(t), \\ \gamma_{022} = -\gamma_{202} &= -B(t) \dot{B}(t), \\ \gamma_{033} = -\gamma_{303} &= -B(t) \dot{B}(t) \sin^2 \theta, \\ \gamma_{233} = -\gamma_{323} &= -B^2(t) \sin \theta \cos \theta. \end{aligned} \quad (4.5)$$

Consequently, The only non-vanishing components of basic vector field are

$$\begin{aligned}\Phi^0 &= -2\left\{\frac{\dot{A}(t)}{A(t)} + \frac{\dot{B}(t)}{B(t)}\right\}, \\ \Phi^2 &= \frac{\cot\theta}{B^2(t)}.\end{aligned}\tag{4.6}$$

Using (4.5) and (4.6) in (4.1) and (4.3), we get

$$E = 0.$$

5 Summary and Discussion

In this paper we have shown that the fourth component of Einstein' complex for the Kantowski-Sachs space-time is not identically zero. This give a counter example to the result obtained by Prasanna [1]. We calculated the total energy of Kantowski-Sachs space-time Using Møller's tetrad theory of gravity. We found that the total energy is zero in this space-time. This result do not agree with the previous results obtained in the both theories of general relativity [10] and teleparallel gravity [12], using Einstein, Bergmann-Thomson and Landau-Lifshitz energy-momentum complexes. In both theories the energy and momentum densities for this space-time are finite and reasonable. We notice that these results are not in conflict with that given by Møller's values for the energy and momentum densities if r tends to infinity.

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